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Generalized Kerr-NUT-de Sitter metrics in all dimensions

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Abstract

We classify all spacetimes with a closed rank-2 conformal Killing-Yano tensor. They give a generalization of Kerr-NUT-de Sitter spacetimes. The Einstein condition is explicitly solved and written as an indefinite integral. It is characterized by a polynomial in the integrand. We briefly discuss the smoothness conditions of the Einstein metrics over compact Riemannian manifolds.

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The D -dimensional Kerr-NUT-de Sitter metric was constructed by Chen-Lü-Pope [1]. The metric is the most general known solution describing the higher-dimensional rotating black hole spacetime with NUT parameters. It takes the form

$$g = \sum_{\mu=1}^n \frac{dx_\mu^2}{Q_\mu(x)} + \sum_{\mu=1}^n Q_\mu(x) \left(\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) d\psi_k \right)^2 + \frac{\varepsilon c}{\sigma_n} \left(\sum_{k=0}^n \sigma_k d\psi_k \right)^2, \quad (1)$$

where $D = 2n + \varepsilon$ ($\varepsilon = 0$ or 1). The functions Q_μ ($\mu = 1, 2, \dots, n$) are given by

$$Q_\mu(x) = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n (x_\mu^2 - x_\nu^2), \quad (2)$$

where $X_\mu = X_\mu(x_\mu)$ is an arbitrary function depending on one coordinate x_μ . The σ_k and $\sigma_k(\hat{x}_\mu)$ are the k -th elementary symmetric functions of $\{x_1^2, \dots, x_n^2\}$ and $\{x_\nu^2 : \nu \neq \mu\}$ respectively:

$$\prod_{\nu=1}^n (t - x_\nu^2) = \sigma_0 t^n - \sigma_1 t^{n-1} + \dots + (-1)^n \sigma_n, \quad (3)$$

$$\prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n (t - x_\nu^2) = \sigma_0(\hat{x}_\mu) t^{n-1} - \sigma_1(\hat{x}_\mu) t^{n-2} + \dots + (-1)^{n-1} \sigma_{n-1}(\hat{x}_\mu). \quad (4)$$

The metric satisfies the Einstein equation $Ric(g) = \Lambda g$ if and only if X_μ takes the form [1, 2],

$$(a) \quad \varepsilon = 0 : X_\mu = \sum_{k=0}^n c_k x_\mu^{2k} + b_\mu x_\mu, \quad (b) \quad \varepsilon = 1 : X_\mu = \sum_{k=0}^n c_k x_\mu^{2k} + b_\mu + \frac{(-1)^n c}{x_\mu^2}, \quad (5)$$

where c, c_k and b_μ are free parameters. This class of metrics gives the Kerr-NUT-de Sitter metric [1], and the solutions in [3, 4, 5, 6, 7] are recovered by choosing special parameters.

It has been shown in [8, 9] that the Kerr-NUT-de Sitter spacetime has a rank-2 closed conformal Killing-Yano (CKY) tensor. This tensor generates the tower of Killing-Yano and Killing tensors, which implies complete integrability of geodesic equations [10] and complete separation of variables for the Hamilton-Jacobi, Klein-Gordon [8, 11] and Dirac equations [16]. Various aspects related to the integrability have been extensively studied in [12, 13, 14, 15, 17, 18, 21, 22]. For reviews on these subjects, see, for example, [19, 20].

This property leads to a natural question whether there are other geometries with such a CKY tensor. The following result was proved in [15].

Theorem 1. Let us assume the existence of a non-degenerate rank-2 CKY tensor h for D -dimensional spacetime (M, g) satisfying the conditions¹,

$$(a1) \, dh = 0, \quad (a2) \, \mathcal{L}_{\hat{\xi}}g = 0, \quad (a3) \, \mathcal{L}_{\hat{\xi}}h = 0.$$

Then, M is only the Kerr-NUT-de Sitter spacetime.

The rank-2 CKY tensor $h = (h_{ab})$ is a 2-form defined by the equation [23]

$$\nabla_a h_{bc} + \nabla_b h_{ac} = 2\hat{\xi}_c g_{ab} - \hat{\xi}_a g_{bc} - \hat{\xi}_b g_{ac}, \quad (6)$$

where the associated vector $\hat{\xi} = \hat{\xi}^a \partial_a$ of h is given by $\hat{\xi}_a = (1/(D-1))\nabla^b h_{ba}$. By introducing the following orthonormal frame

$$e^\mu = \frac{dx_\mu}{\sqrt{Q_\mu}}, \quad e^{n+\mu} = \sqrt{Q_\mu} \sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) d\psi_k, \quad \varepsilon e^{2n+1} = \varepsilon \sqrt{\frac{c}{\sigma_n}} \sum_{k=0}^n \sigma_k d\psi_k \quad (7)$$

for the metric (1), the CKY tensor can be written as

$$h = \sum_{\mu=1}^n x_\mu e^\mu \wedge e^{n+\mu} = d \left(\frac{1}{2} \sum_{k=0}^{n-1} \sigma_{k+1} d\psi_k \right). \quad (8)$$

In [15] we required that the eigenvalues x_μ of h are functionally independent in some spacetime domain, i.e. x_μ are non-constant independent functions. In this paper we do not assume the functional independence of the eigenvalues, and hence the CKY tensor generally has the non-constant eigenvalues and the constant ones. The metric may be locally given as a Kaluza-Klein metric on the bundle over Kähler manifolds whose fibers are Kerr-NUT-de Sitter spacetimes.

¹ Recently, it was proved that the assumptions of (a2) and (a3) are superfluous because they follow from the existence of the closed CKY tensor [22].

Let (M, g) be a D -dimensional spacetime with a closed rank-2 CKY tensor h . Let x_μ ($\mu = 1, \dots, n$) and ξ_i ($i = 1, \dots, N$) be the non-constant eigenvalues and the non-zero constant ones of h , respectively. Suppose the eigenvalues of the “square of the CKY tensor” $Q = (Q^a_b) = (-h^a_c h^c_b)$ have the following multiplicities:

$$\{\underbrace{x_1^2, \dots, x_1^2}_{2\ell_1}, \dots, \underbrace{x_n^2, \dots, x_n^2}_{2\ell_n}, \underbrace{\xi_1^2, \dots, \xi_1^2}_{2m_1}, \dots, \underbrace{\xi_N^2, \dots, \xi_N^2}_{2m_N}, \underbrace{0, \dots, 0}_K\}, \quad (9)$$

where $D = 2(|\ell| + |m|) + K$. Here $|\ell| = \sum_{\mu=1}^n \ell_\mu$ and $|m| = \sum_{i=1}^N m_i$.

Analyses for the non-degenerate and some degenerate cases with $|m| = 0$ can be found in [21].

We can show the following results [24]:

Lemma. It must hold that $\ell_\mu = 1$ for all $\mu = 1, 2, \dots, n$.

Theorem 2. The metric and the CKY tensor take the forms

$$g = \sum_{\mu=1}^n \frac{dx_\mu^2}{P_\mu(x)} + \sum_{\mu=1}^n P_\mu(x) \left(\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) \theta_k \right)^2 + \sum_{i=1}^N \prod_{\mu=1}^n (x_\mu^2 - \xi_i^2) g^{(i)} + \sigma_n g^{(0)}, \quad (10)$$

$$h = \sum_{\mu=1}^n x_\mu dx_\mu \wedge \left(\sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) \theta_k \right) + \sum_{i=1}^N \xi_i \prod_{\mu=1}^n (x_\mu^2 - \xi_i^2) \omega^{(i)}. \quad (11)$$

The metrics $g^{(i)}$ are Kähler metrics on $2m_i$ -dimensional Kähler manifolds $M^{(i)}$ and $\omega^{(i)}$ the corresponding Kähler forms. The metric $g^{(0)}$ is, in general, any metric on a K -dimensional manifold $M^{(0)}$. But if $K = 1$, $g^{(0)}$ can take the special form:

$$\sigma_n g_{\text{special}}^{(0)} = \frac{c}{\sigma_n} \left(\sum_{k=0}^n \sigma_k \theta_k \right)^2. \quad (12)$$

The functions P_μ are defined by

$$P_\mu(x) = \frac{X_\mu}{\prod_{i=1}^N (x_\mu^2 - \xi_i^2)^{m_i} U_\mu}, \quad U_\mu = \prod_{\substack{\nu=1 \\ (\nu \neq \mu)}}^n (x_\mu^2 - x_\nu^2) \quad (13)$$

with an arbitrary function $X_\mu = X_\mu(x_\mu)$ depending on x_μ . The 1-forms θ_k satisfy

$$d\theta_k + 2 \sum_{i=1}^N (-1)^{n-k} \xi_i^{2n-2k-1} \omega^{(i)} = 0, \quad k = 0, 1, \dots, n-1+\varepsilon, \quad (14)$$

where $\varepsilon = 0$ for the general type and $\varepsilon = 1$ for the special type.

Remark 1. Locally, $\omega^{(i)} = dA^{(i)}$, and $\theta_k = d\psi_k - 2 \sum_{i=1}^N (-1)^{n-k} \xi_i^{2(n-k)-1} A^{(i)}$. The closed CKY tensor (11) can be rewritten in a manifestly closed form:

$$h = d \left(\frac{1}{2} \sum_{k=0}^{n-1} \sigma_{k+1} d\psi_k + \sum_{i=1}^N \xi_i \prod_{\mu=1}^n (x_\mu^2 - \xi_i^2) A^{(i)} \right). \quad (15)$$

Remark 2. By sending one of constant eigenvalues, say ξ_N , to zero, the metric of general type smoothly goes to a metric of general type : $N \rightarrow N-1$, $K \rightarrow K+2m_N$, $g^{(0)} \rightarrow g^{(0)} + g^{(N)}$, where $g^{(N)}$ is the Kähler metric. For special type, if we set $c = \xi_N^2$, $\psi_n = \varphi/\xi_N$, and then take the limit², $\xi_N \rightarrow 0$, it goes to a metric of general type : $N \rightarrow N-1$, $K = 1 \rightarrow 2m_N + 1$, $g^{(0)}$ part is given by a Sasakian manifold, i.e., an S^1 -bundle over the Kähler manifold,

$$g_{\text{special}}^{(0)} \rightarrow g^{(N)} + (d\varphi - 2A^{(N)})^2. \quad (16)$$

In the following part, we consider the Einstein condition of the metric (10). We introduce an orthonormal frame $\{e^A\} = \{e^\mu, e^{n+\mu}, e_{(i)}^\alpha, e_{(i)}^{m_i+\alpha}, e_{(0)}^\alpha\}$ on M :

$$\begin{aligned} e^\mu &= \frac{dx_\mu}{\sqrt{P_\mu}}, \quad e^{n+\mu} = \sqrt{P_\mu} \sum_{k=0}^{n-1} \sigma_k(\hat{x}_\mu) \theta_k, \quad e_{(0)}^\alpha = \sqrt{\sigma_n} \hat{e}_{(0)}^\alpha, \\ e_{(i)}^\alpha &= \left(\prod_{\mu=1}^n (x_\mu^2 - \xi_i^2) \right)^{1/2} \hat{e}_{(i)}^\alpha, \quad e_{(i)}^{m_i+\alpha} = \left(\prod_{\mu=1}^n (x_\mu^2 - \xi_i^2) \right)^{1/2} \hat{e}_{(i)}^{m_i+\alpha}, \end{aligned} \quad (17)$$

where $\{\hat{e}_{(i)}^\alpha, \hat{e}_{(i)}^{m_i+\alpha}\}_{\alpha=1,2,\dots,m_i}$ is an orthonormal frame of the Kähler manifold $M^{(i)}$,

$$g^{(i)} = \sum_{\alpha=1}^{m_i} (\hat{e}_{(i)}^\alpha \otimes \hat{e}_{(i)}^\alpha + \hat{e}_{(i)}^{m_i+\alpha} \otimes \hat{e}_{(i)}^{m_i+\alpha}), \quad \omega^{(i)} = \sum_{\alpha=1}^{m_i} \hat{e}_{(i)}^\alpha \wedge \hat{e}_{(i)}^{m_i+\alpha}, \quad (18)$$

² The limit is different from the BPS limit [25, 26]. In this limit, the Sasakian manifold appears as a subspace of base space. While in the BPS limit, the odd dimensional Kerr-NUT-de Sitter space (fiber space) goes to a Sasakian manifold. This BPS limit was first done in [25, 26].

and $\{\hat{e}_{(0)}^\alpha\}_{\alpha=1,2,\dots,K}$ is an orthonormal frame of $M^{(0)}$. For special type, we use

$$e_{(0)}^1 = \sqrt{\frac{c}{\sigma_n}} \sum_{k=0}^n \sigma_k \theta_k, \quad (19)$$

instead of $\{e_{(0)}^\alpha\}$.

The CKY tensor (11) is written as

$$h = \sum_{\mu=1}^n x_\mu e^\mu \wedge e^{n+\mu} + \sum_{i=1}^N \sum_{\alpha=1}^{m_i} \xi_i e_{(i)}^\alpha \wedge e_{(i)}^{m_i+\alpha}. \quad (20)$$

It is convenient to introduce the following scalars

$$P_T^{[k]}(t) := \|(Q - tI)^{-k/2} \hat{\xi}\|^2 = \hat{\xi}_a ((Q - tI)^{-k})^a{}_b \hat{\xi}^b = \sum_{\mu=1}^n \frac{P_\mu}{(x_\mu^2 - t)^k} + \frac{\varepsilon S}{(-t)^k}, \quad (21)$$

$$P_T = \|\hat{\xi}\|^2 = \hat{\xi}_a \hat{\xi}^a = \sum_{\mu=1}^n P_\mu + \varepsilon S = P_T^{[0]}(t), \quad \varepsilon S = \varepsilon \frac{c}{\sigma_n}. \quad (22)$$

The non-zero components of the Ricci tensor of the metric (10) are calculated as

$$\begin{aligned} \mathcal{R}_{\mu,\mu} &= \mathcal{R}_{n+\mu,n+\mu} \\ &= -\frac{1}{2} \frac{\partial^2 P_T}{\partial x_\mu^2} - \sum_{\rho \neq \mu} \frac{1}{x_\rho^2 - x_\mu^2} \left(x_\rho \frac{\partial P_T}{\partial x_\rho} - x_\mu \frac{\partial P_T}{\partial x_\mu} \right) - \frac{\varepsilon}{2x_\mu} \frac{\partial P_T}{\partial x_\mu} \\ &\quad - \sum_{i=1}^N m_i \left(x_\mu \frac{\partial}{\partial x_\mu} + 2 \right) P_T^{[1]}(\xi_i^2), \\ \mathcal{R}_{(\alpha,i),(\beta,i)} &= \mathcal{R}_{(m_i+\alpha,i),(m_i+\beta,i)} \\ &= \frac{\hat{\mathcal{R}}_{\alpha\beta}^{(i)}}{\prod_{\mu=1}^n (x_\mu^2 - \xi_i^2)} - \delta_{\alpha\beta} \left(\sum_{\mu=1}^n x_\mu \frac{\partial}{\partial x_\mu} + (D-1) \right) P_T^{[1]}(\xi_i^2) \\ &\quad - 2\delta_{\alpha\beta} \sum_{\substack{j=1 \\ (j \neq i)}}^N \frac{m_j \xi_j^2}{\xi_i^2 - \xi_j^2} \left(P_T^{[1]}(\xi_i^2) - P_T^{[1]}(\xi_j^2) \right) - 2\delta_{\alpha\beta} (m_i + 1) \xi_i^2 P_T^{[2]}(\xi_i^2), \end{aligned} \quad (23)$$

and for (a) general type ($\varepsilon = 0$)

$$\begin{aligned} \mathcal{R}_{(\alpha,0),(\beta,0)} &= \frac{1}{\sigma_n} \hat{\mathcal{R}}_{\alpha\beta}^{(0)} - \delta_{\alpha\beta} \left(\sum_{\mu=1}^n x_\mu \frac{\partial}{\partial x_\mu} + (D-1) \right) P_T^{[1]}(0) \\ &\quad + 2\delta_{\alpha\beta} \sum_{j=1}^N m_j \left(P_T^{[1]}(0) - P_T^{[1]}(\xi_j^2) \right), \end{aligned} \quad (24)$$

for (b) special type ($K = 1$ and $\varepsilon = 1$)

$$\mathcal{R}_{(1,0),(1,0)} = - \sum_{\rho=1}^n \frac{1}{x_\rho} \frac{\partial P_T}{\partial x_\rho} - 2 \sum_{i=1}^N m_i P_T^{[1]}(\xi_i^2), \quad (25)$$

where $\hat{\mathcal{R}}_{\alpha\beta}^{(i)}$ and $\hat{\mathcal{R}}_{\alpha\beta}^{(0)}$ represent the Ricci components of $g^{(i)}$ and $g^{(0)}$ respectively. Recall that $D = 2n + 2|m| + K$.

Since we are working in an orthonormal frame, the Einstein condition becomes $\mathcal{R}_{AB} = \Lambda \delta_{AB}$. Thus, the Kähler metric $g^{(i)}$ (and the metric $g^{(0)}$ for general type) must be Einstein, i.e. $\hat{\mathcal{R}}_{\alpha\beta}^{(i)} = \lambda^{(i)} \delta_{\alpha\beta}$, ($\hat{\mathcal{R}}_{\alpha\beta}^{(0)} = \lambda^{(0)} \delta_{\alpha\beta}$). We find the following result.

Theorem 3. Let $g^{(i)}$ ($i = 1, \dots, N$) be $2m_i$ -dimensional Kähler-Einstein metrics. Let $g^{(0)}$ be a K -dimensional Einstein metric if it is the general type. Then the metric g is Einstein if and only if X_μ takes the form

$$X_\mu(x_\mu) = x_\mu \left(d_\mu + \int \mathcal{X}(x_\mu) x_\mu^{K-2} \prod_{i=1}^N (x_\mu^2 - \xi_i^2)^{m_i} dx_\mu \right), \quad (26)$$

where

$$\mathcal{X}(x) = \sum_{i=-\varepsilon}^n \alpha_i x^{2i}, \quad \alpha_n = -\Lambda, \quad (27)$$

(a) general ($\varepsilon = 0$),

$$\alpha_0 = (-1)^{n-1} \lambda^{(0)}, \quad (28)$$

(b) special ($K = 1$ and $\varepsilon = 1$),

$$\alpha_0 = (-1)^{n-1} 2c \sum_{j=1}^N \frac{m_j}{\xi_j^2}, \quad \alpha_{-1} = (-1)^{n-1} 2c. \quad (29)$$

Here $\{\alpha_k\}_{k=1,2,\dots,n-1}$ and d_μ are free parameters. (When $K = 0$, $\lambda^{(0)}$ is a free parameter.)

The Einstein constants $\lambda^{(i)}$ of $g^{(i)}$ are given by

$$\lambda^{(i)} = (-1)^{n-1} \mathcal{X}(\xi_i). \quad (30)$$

Remark 3. Note that $x_\mu^\varepsilon X_\mu(x_\mu)$ is a polynomial. (For general type with $K = 1$, because any 1-dimensional metric $g^{(0)}$ is flat, $\alpha_0 = (-1)^{n-1} \lambda^{(0)} = 0$, thus there is no $\log x_\mu$ term.)

Remark 4. As in Remark 2, let us consider the $\xi_N \rightarrow 0$ limit. For general type, it is easy to see that the polynomial X_μ (26) is consistent with the limit. For special type, with $c = \xi_N^2$, $\alpha_0 \rightarrow (-1)^{n-1}2m_N$, $\alpha_{-1}\xi_N^{-2} \rightarrow (-1)^{n-1}2$. The Einstein constants of the Kähler-Einstein metric $g^{(N)}$ and the Einstein metric $g^{(0)}$ (16) are $\lambda^{(N)} = 2(m_N + 1)$ and $\lambda^{(0)} = 2m_N$ respectively. The metric $g^{(0)}$ (16), induced from the special type, is now a Sasaki-Einstein metric.

As a simple example, let us consider the special type metric with $n = 1$ and $N = 1$ for (26). Putting $x = ir$ together with the parameters $\{m_1 = m, \xi_1 = a, c = -a^2, d_1 = (-1)^m 2M\}$ we have

$$P(r) = \left(-\frac{1}{r^2} + \frac{\Lambda}{2(m+1)} \right) (r^2 + a^2) + \frac{2M}{(r^2 + a^2)^m}. \quad (31)$$

The corresponding $2m + 3$ -dimensional metric is given by

$$g^{(2m+3)} = (r^2 + a^2)g^{(2m)} - \frac{dr^2}{P(r)} + P(r)\theta_0^2 + \frac{a^2}{r^2}(\theta_0 - r^2\theta_1)^2. \quad (32)$$

If we choose the Fubini-Study metric on \mathbb{CP}^m for the Einstein-Kähler metric $g^{(2m)}$, we reproduce the Kerr-de Sitter metric with mass M and equal rotation parameter a [5].

Finally, we briefly discuss the Einstein metrics over compact Riemannian manifolds that are obtained from the metric (10). For general values of the parameters in (26) the metrics do not extend smoothly onto compact manifolds. For simplicity we consider an $n = 1$ case of special type (26). Let $(M^{(i)}, g^{(i)}, \omega^{(i)})$ ($i = 1, \dots, N$) be $2m_i$ -dimensional compact Kähler Einstein manifolds with positive first Chern class c_1 . One can write c_1 as $p_i\alpha_i$, where α_i is indivisible and p_i is a positive integer. Let $B = M^{(1)} \times \dots \times M^{(N)}$ and π_i be the projection map onto $M^{(i)}$. We will consider principal T^2 bundles over B which are classified by cohomology classes χ_a ($a = 1, 2$) of the form

$$\chi_a = \sum_{i=1}^N k_i^{(a)} \pi_i^* \alpha_i, \quad k_i^{(a)} \in \mathbb{Z}. \quad (33)$$

Now let us choose the roots x_1 and x_2 to the equation (26), $X(x) = 0$. We take the region $x_1 \leq x \leq x_2$ assuming that $P(x) = X(x)/(x \prod_{i=1}^N (x^2 - \xi_i^2)^{m_i}) \geq 0$. In order to

avoid the singularity at the boundaries $x = x_1$ and x_2 the following quantities must be integers:

$$k_i^{(1)} = \frac{P'(x_1)}{1 - (x_2/x_1)^2} \left(\xi_i - \frac{x_2^2}{\xi_i} \right) \frac{p_i}{\lambda^{(i)}}, \quad k_i^{(2)} = \frac{P'(x_2)}{1 - (x_1/x_2)^2} \left(\xi_i - \frac{x_1^2}{\xi_i} \right) \frac{p_i}{\lambda^{(i)}} \quad (34)$$

for $i = 1, \dots, N$. The integers $k_i^{(a)}$ may be identified with integral coefficients in (33).

For example, we can obtain 5-dimensional Einstein metrics on S^3 -bundle over S^2 constructed in [27] as follows. Let us consider the case $B = \mathbb{CP}^1$. For the real numbers ν_1 and ν_2 we put $\Lambda = 4(1 - \nu_1^2\nu_2^2)/(2 - \nu_1^2 - \nu_2^2)$, $c = \nu_1^2\nu_2^2$ and $\xi_1 = 1$. Then we have

$$P(x) = \frac{(x^2 - \nu_1^2)(x^2 - \nu_2^2)(1 - \Lambda x^2/4)}{x^2(x^2 - 1)}. \quad (35)$$

The integral condition (34) is written as

$$k^{(1)} = \frac{\nu_1(1 - \nu_2^2)(2 - \nu_2^2 - \nu_1^2\nu_2^2)}{1 + \nu_1^4\nu_2^2 + \nu_1^2\nu_2^4 - 3\nu_1^2\nu_2^2}, \quad k^{(2)} = \frac{\nu_2(1 - \nu_1^2)(2 - \nu_1^2 - \nu_1^2\nu_2^2)}{1 + \nu_1^4\nu_2^2 + \nu_1^2\nu_2^4 - 3\nu_1^2\nu_2^2}, \quad (36)$$

which is just the condition for the existence of Einstein metrics given in [27].

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